

Local Hidden Variable Theoretic Measure of Quantumness of Mutual Information *

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1 Introduction

Entanglement, a manifestation of quantumness of correlations between observables of the subsystems of a composite system, once thought to be an essential ingredient for distinctive quantum features in quantum information processing, is no longer considered to be so as it is found that the unique features of quantum information processing are contained in the quantum nature of mutual information which does not necessarily require entanglement (see [1] and references therein). Whereas the concept of quantumness of correlations between observables of the parts of a composite system is characterized by their incommensurability with the predictions of the local hidden variable (LHV) theory (see [2] and references therein), that of the quantumness of information does not invoke the LHV theory [3]-[6] explicitly. Different protocols for identifying classical content of information lead to different measures of quantumness of mutual information like quantum discord [3], quantum deficit [4], measurement induced disturbance [5], symmetric discord [6] and others [1]. A number of analytic and numerical results for these measures of quantumness for various states of two qubits have been reported [3]-[12]. These results show that even a separable state may contain quantum features in its information content.

In this paper a measure of quantumness of mutual information is proposed by invoking the LHV theory explicitly. The proposed measure turns out to be useful as it circumvents the need of optimization of classical information over possible directions of measurement for a class of states and simplifies finding optimized classical information for others. Moreover, under specific situations, it fits in with one or the other widely used measures, namely, the measurement induced disturbance, the symmetric discord, and the quantum discord.

To that end, the classical mutual information I_{LHV} in this paper is defined following the LHV theoretic considerations of [13] regarding characterization

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of quantumness of correlations between observables in a system of spins A and B described by the density operator $\hat{\rho}^{AB}$. It is used to measure the quantumness of mutual information as $Q_{LHV} = I_Q(\hat{\rho}^{AB}) - I_{LHV}$, where $I_Q(\hat{\rho}^{AB})$ is the quantum information in $\hat{\rho}^{AB}$. The Q_{LHV} is found to be identical with the measurement induced disturbance if the Bloch vectors $\langle \hat{\mathbf{S}}^A \rangle$ and $\langle \hat{\mathbf{S}}^B \rangle$ of spins A and B are non-zero where $\hat{\mathbf{S}}^A$ ($\hat{\mathbf{S}}^B$) is the spin vector of spin A (spin B) and $\langle \hat{P} \rangle = \text{Tr}(\hat{P}\hat{\rho}^{AB})$. If $\langle \hat{\mathbf{S}}^A \rangle = \langle \hat{\mathbf{S}}^B \rangle = 0$ then I_{LHV} is the maximum value of classical mutual information over directions of measurement of the two spins which can be evaluated analytically exactly. The Q_{LHV} then turns out to be the same as the symmetric discord. If one of the Bloch vectors, say, $\langle \hat{\mathbf{S}}^A \rangle = 0$, but the other is not then, for certain states, Q_{LHV} is same as the quantum discord for measurement over A. Thus the LHV theoretic quantumness of mutual information and the measurement induced disturbance are identical when the Bloch vector of each spin is non-zero. However, whereas the measurement induced disturbance is non-unique when the Bloch vector of either or both the spins is zero, the LHV theoretic measure determines the quantumness of mutual information uniquely even in those situations.

The paper is organized as follows. The section 2 presents the formulation of LHV theoretic quantumness of mutual information. It is compared with other measures in section 3. The conclusions are summarized in section 4.

2 Local Hidden Variable Theory and Quantumness

Let us recall that a spin-1/2 in LHV theory is regarded as a vector \mathbf{S} in the real three dimensional space whose component along any direction can assume two values, say $\pm 1/2$, and is assumed to be under the influence of some unknown *hidden* causes or variables acting randomly. The random influence of the hidden variables results in the components of the spin in any direction acquiring randomly the values $\pm 1/2$. The properties of the spin may then be described in terms of the probability distribution functions $f(S_{a_1}, S_{a_2}, \dots, S_{a_N})$ for the components of the spin in the directions $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N$ where

$$S_{a_i} = \mathbf{S} \cdot \mathbf{a}_i, \quad |\mathbf{a}_i| = 1, \quad (1)$$

is the component of spin in the direction \mathbf{a}_i . Now, let $p(\epsilon_{a_1}, \epsilon_{a_2}, \dots, \epsilon_{a_N})$ ($\epsilon_{a_i} = \pm 1$) denote the joint probability for the components of the spin along the directions $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N$ to have the values $\epsilon_{a_1}/2, \epsilon_{a_2}/2, \dots, \epsilon_{a_N}/2$ respectively so that

$$\begin{aligned} & f(S_{a_1}, S_{a_2}, \dots, S_{a_N}) \\ &= \sum_{\epsilon_{a_1}, \epsilon_{a_2}, \dots, \epsilon_{a_N} = \pm 1} \left[\delta\left(S_{a_1} - \frac{\epsilon_{a_1}}{2}\right) \delta\left(S_{a_2} - \frac{\epsilon_{a_2}}{2}\right) \dots \delta\left(S_{a_N} - \frac{\epsilon_{a_N}}{2}\right) \right] \\ & \quad \times p(\epsilon_{a_1}, \epsilon_{a_2}, \dots, \epsilon_{a_N}). \end{aligned} \quad (2)$$

It is straightforward to invert this relation to get

$$p(\epsilon_{a_1}, \epsilon_{a_2}, \dots, \epsilon_{a_N}) = \left\langle \left(\frac{1}{2} + \epsilon_{a_1} S_{a_1} \right) \left(\frac{1}{2} + \epsilon_{a_2} S_{a_2} \right) \cdots \left(\frac{1}{2} + \epsilon_{a_N} S_{a_N} \right) \right\rangle, \quad (3)$$

where the angular bracket denotes average with respect to the distribution function $f(S_{a_1}, S_{a_2}, \dots, S_{a_N})$:

$$\langle P \rangle = \int P f(S_{a_1}, S_{a_2}, \dots, S_{a_N}) \prod_{i=1}^N dS_{a_i}. \quad (4)$$

The joint probability distribution for two spins can be similarly defined and shown to be expressible as

$$\begin{aligned} p(\epsilon_{a_1}^A, \epsilon_{a_2}^A, \dots, \epsilon_{a_M}^A; \epsilon_{b_1}^B, \epsilon_{b_2}^B, \dots, \epsilon_{b_N}^B) \\ = \left\langle \left(\frac{1}{2} + \epsilon_{a_1}^A S_{a_1} \right) \left(\frac{1}{2} + \epsilon_{a_2}^A S_{a_2} \right) \cdots \left(\frac{1}{2} + \epsilon_{a_M}^A S_{a_M} \right) \right. \\ \left. \left(\frac{1}{2} + \epsilon_{b_1}^B S_{b_1} \right) \left(\frac{1}{2} + \epsilon_{b_2}^B S_{b_2} \right) \cdots \left(\frac{1}{2} + \epsilon_{b_N}^B S_{b_N} \right) \right\rangle, \end{aligned} \quad (5)$$

where $p(\epsilon_{a_1}^A, \epsilon_{a_2}^A, \dots, \epsilon_{a_M}^A; \epsilon_{b_1}^B, \epsilon_{b_2}^B, \dots, \epsilon_{b_N}^B)$ stands for the probability of finding the components of spin A to have values $\epsilon_{a_1}^A/2, \epsilon_{a_2}^A/2, \dots, \epsilon_{a_M}^A/2$ in the directions $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_M$, and the components of spin B to have the values $\epsilon_{b_1}^B/2, \epsilon_{b_2}^B/2, \dots, \epsilon_{b_N}^B/2$ in the directions $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_N$ with $\epsilon_{a_i}^A, \epsilon_{b_i}^B = \pm 1$. The form (5) for the joint probability is useful for constructing its quantum analog by (i) replacing the classical random variables S_a by the operators \hat{S}_a which obey the commutation relation

$$[\hat{S}_a, \hat{S}_b] = i(\mathbf{a} \times \mathbf{b}) \cdot \hat{\mathbf{S}}, \quad (6)$$

where $\hat{\mathbf{S}} = \mathbf{e}_1 \hat{S}_{e_1} + \mathbf{e}_2 \hat{S}_{e_2} + \mathbf{e}_3 \hat{S}_{e_3}$ ($\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$), and the anti-commutation relation

$$\hat{S}_a \hat{S}_b + \hat{S}_b \hat{S}_a = \frac{\mathbf{a} \cdot \mathbf{b}}{2}, \quad (7)$$

(ii) by assigning a rule, called the Chosen Ordering, for ordering operators in a product of non-commuting operators, and (iii) by replacing the average therein as the quantum mechanical expectation value wherein the system is described by a density matrix $\hat{\rho}$ and the expectation value of an operator \hat{P} is given by $\langle \hat{P} \rangle = \text{Tr}(\hat{P} \hat{\rho})$. This approach has been used in [13] to formulate a criterion for identifying states admitting LHV description.

Now, let $\hat{\rho}^{AB}$ describe the state of a system of two spin-1/2 particles, A and B. Following the approach outlined above, the expression for the joint probability $p(\epsilon_a^A, \epsilon_b^B)$ for the component S_a of spin A in the direction \mathbf{a} to have value $\epsilon_a^A/2$, and the component S_b of spin B in direction \mathbf{b} to have the value $\epsilon_b^B/2$ ($\epsilon_a^A, \epsilon_b^B = \pm 1$) may be seen to be given by,

$$p(\epsilon_a^A, \epsilon_b^B) = \text{Tr} \left[\left(\frac{1}{2} + \epsilon_a^A \hat{S}_a \right) \left(\frac{1}{2} + \epsilon_b^B \hat{S}_b \right) \hat{\rho}^{AB} \right]. \quad (8)$$

In this case the issue of operator ordering does not arise as there are no non-commuting operators in the product in the expression above. The corresponding marginal distributions are

$$\begin{aligned} p(\epsilon_a^A) &= \text{Tr} \left[\left(\frac{1}{2} + \epsilon_a^A \hat{S}_a^A \right) \hat{\rho}^{AB} \right] \equiv \sum_{\epsilon_b^B} p(\epsilon_a^A, \epsilon_b^B), \\ p(\epsilon_b^B) &= \text{Tr} \left[\left(\frac{1}{2} + \epsilon_b^B \hat{S}_b^B \right) \hat{\rho}^{AB} \right] \equiv \sum_{\epsilon_a^A} p(\epsilon_a^A, \epsilon_b^B). \end{aligned} \quad (9)$$

The mutual information corresponding to $p(\epsilon_a^A, \epsilon_b^B)$ in (8) is

$$I(\mathbf{a}, \mathbf{b}) = S(p(\epsilon_a^A)) + S(p(\epsilon_b^B)) - S(p(\epsilon_a^A, \epsilon_b^B)), \quad (10)$$

where $S(p(\{x_i\}_n))$ is Shannon entropy for the probability $p(x_1, x_2, \dots, x_n) \equiv p(\{x_i\}_n)$ of n random variables x_1, x_2, \dots, x_n :

$$S(p(\{x_i\}_n)) = - \sum_{\{x_i\}_n} p(\{x_i\}_n) \log p(\{x_i\}_n), \quad (11)$$

and the logarithm is to the base 2. The Eq.(10) gives the LHV theoretic classical expression for information in the distribution function of the components in directions \mathbf{a} and \mathbf{b} .

On the other hand, the quantum theoretic mutual information for the system described by the density operator $\hat{\rho}^{AB}$ is given by

$$I_Q(\hat{\rho}^{AB}) = S_Q(\hat{\rho}^A) + S_Q(\hat{\rho}^B) - S_Q(\hat{\rho}^{AB}), \quad (12)$$

where $S_Q(\hat{\rho})$ denotes the von Neumann entropy:

$$S_Q(\hat{\rho}) = -\text{Tr}[\hat{\rho} \log(\hat{\rho})], \quad (13)$$

and $\hat{\rho}^A = \text{Tr}_B(\hat{\rho}^{AB})$, $\hat{\rho}^B = \text{Tr}_A(\hat{\rho}^{AB})$ are the reduced density operators of the spins A and B respectively.

The LHV theoretic quantumness of mutual information for the joint probability for the component of A in the direction \mathbf{a} and that of B in the direction \mathbf{b} to have the values $\pm 1/2$ may be defined as

$$Q(\mathbf{a}, \mathbf{b}) = I_Q(\hat{\rho}^{AB}) - I(\mathbf{a}, \mathbf{b}). \quad (14)$$

Different measures of quantumness are obtained by different choices of the directions \mathbf{a} and \mathbf{b} . It is proposed to specify \mathbf{a} and \mathbf{b} by noting the following:

1. The variance in the measurement of $\hat{\mathbf{S}} \cdot \mathbf{b}$ i.e. in the component of spin along the direction \mathbf{b} is given by

$$\Delta(\hat{\mathbf{S}} \cdot \mathbf{b})^2 = \langle (\hat{\mathbf{S}} \cdot \mathbf{b})^2 \rangle - \langle \hat{\mathbf{S}} \cdot \mathbf{b} \rangle^2 = \frac{1}{4} - \langle (\hat{\mathbf{S}}) \cdot \mathbf{b} \rangle^2. \quad (15)$$

This shows that the variance is minimum when \mathbf{b} is in the direction of $\langle \hat{\mathbf{S}} \rangle$ i.e. in the direction of the Bloch vector of the spin.

2. Let $|\pm, \mathbf{a}\rangle$ denote the eigenstates of the spin component $\hat{\mathbf{S}} \cdot \mathbf{a}$ in the direction \mathbf{a} . Let $\hat{\rho}$ be the density matrix describing the state of the spin-1/2 particle and let $\hat{\rho}_a$ denote its state after measurement of its component along the direction \mathbf{a} . It can be shown that $S(\hat{\rho}) \leq S(\hat{\rho}_a)$ with equality holding if and only if \mathbf{a} is such that $|\pm, \mathbf{a}\rangle$ is the eigenstates of $\hat{\rho}$ [5]. Thus the least disturbing measurement is along the direction \mathbf{a} which is such that $|\pm, \mathbf{a}\rangle$ are the eigenstates of $\hat{\rho}$. Now, recall that the density matrix of a spin-1/2 particle may be written as

$$\hat{\rho} = \frac{I}{2} + 2\langle \hat{\mathbf{S}} \rangle \cdot \hat{\mathbf{S}}. \quad (16)$$

This shows that the eigenstates of the spin component along the direction $\langle \hat{\mathbf{S}} \rangle$ are the eigenstates of $\hat{\rho}$ as well. Hence the least disturbing measurement in the sense described above is along the direction of $\langle \hat{\mathbf{S}} \rangle$.

We thus see that the direction of $\langle \hat{\mathbf{S}} \rangle$ i.e. the direction of the Bloch vector has special significance as the one in which the spin component has minimum variance and also the one along which the measurement is least disturbing. Note also that the criterion for identifying quantumness in the correlations between observables in [13] is based on the properties of the joint quasiprobability in symmetric ordering for the eigenvalues of the components of each spin in three mutually orthogonal directions, one of which is the direction of the Bloch vector of that spin, and on the said quasiprobability for two of the three said components. That criterion identifies non-classical states of two or more spin-1/2 particles in agreement with the predictions based on other approaches, including the prediction of classicality of certain non-separable states.

In view of the discussion above, we let \mathbf{a} and \mathbf{b} in (14) to be the directions of the Bloch vectors of spins A and B respectively if those vectors are non-zero and define the quantumness of mutual information as [14]

$$Q_{\text{LHV}} = I_Q(\hat{\rho}^{\text{AB}}) - I_{\text{LHV}}, \quad (17)$$

where

$$I_{\text{LHV}} \equiv I(\mathbf{a}, \mathbf{b}), \quad \mathbf{a} = \frac{\langle \hat{\mathbf{S}}^{\text{A}} \rangle}{|\langle \hat{\mathbf{S}}^{\text{A}} \rangle|} \neq 0, \quad \mathbf{b} = \frac{\langle \hat{\mathbf{S}}^{\text{B}} \rangle}{|\langle \hat{\mathbf{S}}^{\text{B}} \rangle|} \neq 0. \quad (18)$$

We will see that Q_{LHV} in this case is identical with the measurement induced disturbance. However, the measurement induced disturbance does not specify \mathbf{a} (\mathbf{b}) uniquely when $\langle \hat{\mathbf{S}}^{\text{A}} \rangle = 0$ ($\langle \hat{\mathbf{S}}^{\text{B}} \rangle = 0$) but, as discussed below, I_{LHV} can be specified and Q_{LHV} determined uniquely even in such cases.

1. Let $\langle \hat{\mathbf{S}}^{\text{A}} \rangle = 0$ but $\langle \hat{\mathbf{S}}^{\text{B}} \rangle \neq 0$. The \mathbf{a} in (18) can then be any direction. In that case I_{LHV} is defined to be the maximum of $I(\mathbf{a}, \mathbf{b})$ over all \mathbf{a} :

$$I_{\text{LHV}} = \max_{\mathbf{a}} I(\mathbf{a}, \mathbf{b}), \quad \langle \hat{\mathbf{S}}^{\text{A}} \rangle = 0, \quad \mathbf{b} = \frac{\langle \hat{\mathbf{S}}^{\text{B}} \rangle}{|\langle \hat{\mathbf{S}}^{\text{B}} \rangle|} \neq 0. \quad (19)$$

We will see that Q_{LHV} in this case is the same as quantum discord if $\hat{\rho}^{\text{AB}}$ satisfies the condition specified following Eq.(44).

2. If $\langle \hat{\mathbf{S}}^{\text{A}} \rangle = \langle \hat{\mathbf{S}}^{\text{B}} \rangle = 0$, i.e. $\langle \hat{S}_a^{\text{A}} \rangle = \langle \hat{S}_b^{\text{B}} \rangle = 0$ for all \mathbf{a} and \mathbf{b} then both, \mathbf{a} and \mathbf{b} , in (18) are arbitrary. In this case I_{LHV} is defined to be the maximum of $I(\mathbf{a}, \mathbf{b})$ over all directions \mathbf{a} and \mathbf{b} :

$$I_{\text{LHV}} = \max_{\mathbf{a}, \mathbf{b}} I(\mathbf{a}, \mathbf{b}), \quad \langle \hat{\mathbf{S}}^{\text{A}} \rangle = \langle \hat{\mathbf{S}}^{\text{B}} \rangle = 0. \quad (20)$$

By evaluating the expression above analytically exactly in the following, we will show that Q_{LHV} in this case is the same as the symmetric discord.

To that end, let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal set of cartesian vectors and let $\mathbf{a} = \sum_i u_i \mathbf{e}_i$, $\mathbf{b} = \sum_i v_i \mathbf{e}_i$ $\hat{\mathbf{S}}^\mu = \sum_i \mathbf{e}_i \hat{S}_i^\mu$ ($\mu = \text{A}, \text{B}$) so that

$$\hat{S}_a^{\text{A}} = \mathbf{a} \cdot \hat{\mathbf{S}}^{\text{A}} = \sum_{i=1}^3 u_i \hat{S}_i^{\text{A}}, \quad \hat{S}_b^{\text{B}} = \mathbf{b} \cdot \hat{\mathbf{S}}^{\text{B}} = \sum_{i=1}^3 v_i \hat{S}_i^{\text{B}}, \quad (21)$$

and

$$\langle \hat{S}_a^{\text{A}} \hat{S}_b^{\text{B}} \rangle = \frac{1}{4} \sum_{i,j=1}^3 C_{ij} u_i v_j, \quad (22)$$

where

$$C_{ij} = 4 \text{Tr} \left(\hat{S}_i^{\text{A}} \hat{S}_j^{\text{B}} \hat{\rho}^{\text{AB}} \right), \quad (23)$$

denotes correlation between the components of the spins A and B. The expression (8) for $p(\epsilon_a^{\text{A}}; \epsilon_b^{\text{B}})$ then assumes the form

$$\begin{aligned} p(\epsilon_a^{\text{A}}; \epsilon_b^{\text{B}}) &= \text{Tr} \left[\left(\frac{1}{4} + \epsilon_a^{\text{A}} \epsilon_b^{\text{B}} \hat{S}_a^{\text{A}} \hat{S}_b^{\text{B}} \right) \hat{\rho}^{\text{AB}} \right] \\ &= \frac{1}{4} \left(1 + \epsilon_a^{\text{A}} \epsilon_b^{\text{B}} \sum_{i,j=1}^3 C_{ij} u_i v_j \right), \end{aligned} \quad (24)$$

and $p(\epsilon_a^{\text{A}}) = p(\epsilon_b^{\text{B}}) = 1/2$. It can be shown that in this case [6]

$$\max_{\mathbf{a}, \mathbf{b}} I(\mathbf{a}, \mathbf{b}) = 1 + H \left(\frac{1+C}{2}, \frac{1-C}{2} \right), \quad (25)$$

with

$$H(\{x_i\}_N) = - \sum_{i=1}^N x_i \log(x_i), \quad (26)$$

and

$$C = \max(|f_1|, |f_2|, |f_3|), \quad (27)$$

where (f_1, f_2, f_3) are the singular values of the matrix $\hat{C} \equiv \{C_{ij}\}$. It then follows that

$$Q_{\text{LHV}} = I_Q(\hat{\rho}^{\text{AB}}) - \left[1 + H\left(\frac{1+C}{2}, \frac{1-C}{2}\right) \right]. \quad (28)$$

This is an exact analytic expression for the LHV theoretic quantumness of mutual information when the Bloch vector of each spin vanishes.

3 Comparison with Other Measures

Let us now compare the measure introduced above with other measures. To that end, let $\prod_{\pm\mathbf{a}}^{\text{A}}$ and $\prod_{\pm\mathbf{b}}^{\text{B}}$ be the complete set of one-dimensional orthogonal projection operators for projective measurements in directions \mathbf{a} and \mathbf{b} on spins A and B. The state of the system after the measurement would be [5]

$$\hat{\rho}^{\text{AB}}(\mathbf{a}, \mathbf{b}) = \sum_{\mu, \nu = \pm 1} \prod_{\mu\mathbf{a}}^{\text{A}} \otimes \prod_{\nu\mathbf{b}}^{\text{B}} \hat{\rho}^{\text{AB}} \prod_{\mu\mathbf{a}}^{\text{A}} \otimes \prod_{\nu\mathbf{b}}^{\text{B}}. \quad (29)$$

By noting that

$$\prod_{\pm\mathbf{a}}^{\text{A}} = \frac{1}{2} \pm \hat{S}_a^{\text{A}}, \quad \prod_{\pm\mathbf{b}}^{\text{B}} = \frac{1}{2} \pm \hat{S}_b^{\text{B}}, \quad (30)$$

it is straightforward to see that

$$I_Q(\hat{\rho}^{\text{AB}}(\mathbf{a}, \mathbf{b})) = I(\mathbf{a}, \mathbf{b}). \quad (31)$$

In the following we use the results above for comparing the LHV theoretic and other measures of quantumness of mutual information.

1. Consider the measurement induced disturbance measure defined as

$$Q_{\text{MID}} = I_Q(\hat{\rho}^{\text{AB}}) - I_Q(\hat{\rho}^{\text{AB}}(\mathbf{a}, \mathbf{b})), \quad (32)$$

where $\prod_{\pm\mathbf{a}}^{\text{A}}$ and $\prod_{\pm\mathbf{b}}^{\text{B}}$ are projections on the eigenbasis of the reduced density operators $\hat{\rho}^{\text{A}}$ and $\hat{\rho}^{\text{B}}$ of spins A and B respectively. A reason for the choice of measurement induced by projectors on the eigenbases of the density operators is that such measurements are least disturbing [5]. As shown following Eq.(16), in this case \mathbf{a} and \mathbf{b} are also the directions of $\langle \hat{\mathbf{S}}^{\text{A}} \rangle$ and $\langle \hat{\mathbf{S}}^{\text{B}} \rangle$. Thus, $\prod_{\pm\mathbf{a}}$ are projectors on the eigenbasis of $\hat{\rho}^{\text{A}}$ and \mathbf{a} is also the direction of the Bloch vector of spin A with similar observation about the spin B. From (31) and (18) it then follows that

$$I_Q(\hat{\rho}(\mathbf{a}, \mathbf{b})) = I_{\text{LHV}}, \quad \mathbf{a} = \frac{\langle \hat{\mathbf{S}}^{\text{A}} \rangle}{|\langle \hat{\mathbf{S}}^{\text{A}} \rangle|} \neq 0, \quad \mathbf{b} = \frac{\langle \hat{\mathbf{S}}^{\text{B}} \rangle}{|\langle \hat{\mathbf{S}}^{\text{B}} \rangle|} \neq 0. \quad (33)$$

Hence, the measurement induced disturbance and LHV theoretic measures are same when the Bloch vectors of the two spins are non-zero:

$$Q_{\text{LHV}} = Q_{\text{MID}}, \quad \mathbf{a} = \frac{\langle \hat{\mathbf{S}}^A \rangle}{|\langle \hat{\mathbf{S}}^A \rangle|} \neq 0, \quad \mathbf{b} = \frac{\langle \hat{\mathbf{S}}^B \rangle}{|\langle \hat{\mathbf{S}}^B \rangle|} \neq 0. \quad (34)$$

In case $\langle \hat{\mathbf{S}}^A \rangle = 0$, (16) shows that $\hat{\rho}^A = I/2$ which means that the eigenbasis of reduced density operator of A is not unique. In such cases, measurement induced disturbance is not unique whereas Q_{LHV} , evaluated as in (19), is uniquely determined and will be shown below to be analogous to quantum discord if $\hat{\rho}^{\text{AB}}$ satisfies the condition specified following Eq.(44).

2. The quantum discord for projective measurement on A is defined as [15]

$$Q_{\text{D}} = S_{\text{Q}}(\hat{\rho}^A) - S_{\text{Q}}(\hat{\rho}^{\text{AB}}) + \min_{\mathbf{a}} \sum_{\mu=\pm} p_{\mu} S_{\text{Q}}(\hat{\rho}_{\mu}), \quad (35)$$

where

$$p_{\mu} = \text{Tr} \hat{\Pi}_{\mu\mathbf{a}}^A \hat{\rho}^{\text{AB}} \hat{\Pi}_{\mu\mathbf{a}}^A, \quad \hat{\rho}_{\mu} = \frac{\hat{\Pi}_{\mu\mathbf{a}}^A \hat{\rho}^{\text{AB}} \hat{\Pi}_{\mu\mathbf{a}}^A}{p_{\mu}}. \quad (36)$$

We have

$$\begin{aligned} \sum_{\mu} p_{\mu} S_{\text{Q}}(\hat{\rho}_{\mu}) &= S_{\text{Q}}(\hat{\rho}'^{\text{AB}}(\mathbf{a})) - S_{\text{Q}}(\hat{\rho}'^A(\mathbf{a})) \\ &= S_{\text{Q}}(\hat{\rho}_{++}^{\text{B}}(\mathbf{a})) + S_{\text{Q}}(\hat{\rho}_{--}^{\text{B}}(\mathbf{a})) - S_{\text{Q}}(\hat{\rho}'^A(\mathbf{a})), \end{aligned} \quad (37)$$

where

$$\hat{\rho}'^{\text{AB}}(\mathbf{a}) = \sum_{\mu=\pm} \hat{\Pi}_{\mu\mathbf{a}}^A \hat{\rho}^{\text{AB}} \hat{\Pi}_{\mu\mathbf{a}}^A, \quad \hat{\rho}'^A(\mathbf{a}) = \text{Tr}_{\text{B}}(\hat{\rho}'^{\text{AB}}(\mathbf{a})), \quad (38)$$

and

$$\hat{\rho}_{\mu\mu}^{\text{B}}(\mathbf{a}) = \langle \mu\mathbf{a} | \hat{\rho}^{\text{AB}} | \mu\mathbf{a} \rangle, \quad \mu = \pm. \quad (39)$$

Now, let optimization in (35) be attained for $\mathbf{a} = \mathbf{a}_m$ then, on invoking (37), the last term in (35) assumes the form

$$\begin{aligned} \min_{\mathbf{a}} \sum_{\mu=\pm} p_{\mu} S_{\text{Q}}(\hat{\rho}_{\mu}) &= S_{\text{Q}}(\hat{\rho}_{++}^{\text{B}}(\mathbf{a}_m)) + S_{\text{Q}}(\hat{\rho}_{--}^{\text{B}}(\mathbf{a}_m)) \\ &\quad - S_{\text{Q}}(\hat{\rho}'^A(\mathbf{a}_m)). \end{aligned} \quad (40)$$

If \mathbf{a}_m is such that the eigenbasis of $\hat{\rho}_{++}^{\text{B}}(\mathbf{a}_m)$ and that of $\hat{\rho}_{--}^{\text{B}}(\mathbf{a}_m)$ is the same as the eigenbasis $|\pm \mathbf{b}\rangle$ of $\hat{\rho}^{\text{B}}$ then

$$S_{\text{Q}}(\hat{\rho}_{++}^{\text{B}}(\mathbf{a}_m)) + S_{\text{Q}}(\hat{\rho}_{--}^{\text{B}}(\mathbf{a}_m))$$

$$\begin{aligned}
&= H(\langle \mathbf{b} | \hat{\rho}_{++}^B | \mathbf{b} \rangle, \langle -\mathbf{b} | \hat{\rho}_{++}^B | -\mathbf{b} \rangle) \\
&\quad + H(\langle \mathbf{b} | \hat{\rho}_{--}^B | \mathbf{b} \rangle, \langle -\mathbf{b} | \hat{\rho}_{--}^B | -\mathbf{b} \rangle) \\
&= H(\langle \mathbf{a}_m, \mathbf{b} | \hat{\rho}^{AB} | \mathbf{a}_m, \mathbf{b} \rangle, \langle \mathbf{a}_m, -\mathbf{b} | \hat{\rho}^{AB} | \mathbf{a}_m, -\mathbf{b} \rangle) \\
&\quad + H(\langle -\mathbf{a}_m, \mathbf{b} | \hat{\rho}^{AB} | -\mathbf{a}_m, \mathbf{b} \rangle, \langle -\mathbf{a}_m, -\mathbf{b} | \hat{\rho}^{AB} | -\mathbf{a}_m, -\mathbf{b} \rangle) \\
&= S(p(\epsilon_{a_m}^A, \epsilon_b^B)), \tag{41}
\end{aligned}$$

where $p(\epsilon_{a_m}^A, \epsilon_b^B)$ is as in (8). Also, if $\langle \hat{\mathbf{S}}^A \rangle = 0$ then $\hat{\rho}^A = \hat{\rho}'^A = I$. The expression (35) for the quantum discord may then be rewritten as

$$Q_D = I_Q(\hat{\rho}^{AB}) - I(\mathbf{a}_m, \mathbf{b}). \tag{42}$$

This may be interpreted as

$$Q_D = I_Q(\hat{\rho}^{AB}) - \max_{\mathbf{a}} I(\mathbf{a}, \mathbf{b}). \tag{43}$$

By virtue of (19), the right hand side of this expression is Q_{LHV} . Thus we find that

$$Q_{\text{LHV}} = Q_D, \quad \langle \hat{\mathbf{S}}^A \rangle = 0, \quad \langle \hat{\mathbf{S}}^B \rangle \neq 0. \tag{44}$$

It should be emphasized that the result above is valid only when the eigenbases of $\hat{\rho}_{\pm, \pm}^B(\mathbf{a}_m)$ in the optimum measurement are same as the eigenbasis of $\hat{\rho}^B$.

3. Next, recall that the symmetric discord Q_{SYM} is defined by

$$Q_{\text{SYM}} = I_Q(\hat{\rho}^{AB}) - \max_{\mathbf{a}, \mathbf{b}} I_Q(\hat{\rho}^{AB}), \tag{45}$$

where $\hat{\rho}^{AB}$ is as in (29). By recalling (31) it follows that

$$Q_{\text{SYM}} = I_Q(\hat{\rho}^{AB}) - \max_{\mathbf{a}, \mathbf{b}} I(\mathbf{a}, \mathbf{b}), \tag{46}$$

which, on invoking (20), yields

$$Q_{\text{LHV}} = Q_{\text{SYM}}, \quad \langle \hat{\mathbf{S}}^A \rangle = \langle \hat{\mathbf{S}}^B \rangle = 0. \tag{47}$$

Thus Q_{SYM} is same as Q_{LHV} if the Bloch vector of each spins is zero.

As examples, consider first the pure state. The Bloch vector of each spin in this case is non-zero. By virtue of the considerations above, it follows that Q_{LHV} then is same as the measurement induced disturbance. It turns out to be the same also as the symmetric and the quantum discords.

As regards mixed states, recall that any mixed state of two qubits can be expressed as [7]

$$\hat{\rho}^{AB} = \frac{1}{4} I^A \otimes I^B + \sum_{i,j=1,2,3} w_{ij} \hat{S}_i^A \otimes \hat{S}_j^B + \frac{r}{2} \hat{S}_3^A \otimes I^B + \frac{s}{2} I^A \otimes \hat{S}_3^B. \tag{48}$$

Here $\hat{S}_i^\mu = \mathbf{e}_i \cdot \hat{\mathbf{S}}^\mu$ with $\mu = A, B$ and $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. The optimization involved in symmetric and other discords is generally a formidable task. We consider some special cases.

1. Consider the case $r = s = 0$. The expression (48) then reads

$$\hat{\rho}^{\text{AB}} = \frac{1}{4}I^{\text{A}} \otimes I^{\text{B}} + \sum_{i,j=1,2,3} w_{ij} \hat{S}_i^{\text{A}} \otimes \hat{S}_j^{\text{B}}, \quad (49)$$

so that

$$\langle \hat{\mathbf{S}}^{\text{A}} \rangle = \langle \hat{\mathbf{S}}^{\text{B}} \rangle = 0, \quad \langle \hat{S}_i^{\text{A}} \hat{S}_j^{\text{B}} \rangle = \frac{w_{ij}}{4}. \quad (50)$$

The exact expression for Q_{LHV} in this case is given by (28). Insert in it the expression for $I_Q(\hat{\rho}^{\text{AB}})$ with $\hat{\rho}^{\text{AB}}$ given by (49). It can be shown that

$$I_Q(\hat{\rho}^{\text{AB}}) = 2 - H(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \quad (51)$$

where λ_i ($i = 1, 2, 3, 4$) are the eigenvalues of $\hat{\rho}^{\text{AB}}$ given by

$$\begin{aligned} \lambda_1 &= \frac{1 - f_1 - f_2 - f_3}{4}, & \lambda_2 &= \frac{1 - f_1 + f_2 + f_3}{4}, \\ \lambda_3 &= \frac{1 + f_1 - f_2 + f_3}{4}, & \lambda_4 &= \frac{1 + f_1 + f_2 - f_3}{4}, \end{aligned} \quad (52)$$

with f_1, f_2, f_3 being the eigenvalues of the 3×3 matrix formed by w_{ij} ($i, j = 1, 2, 3$) as its elements. Hence

$$Q_{\text{LHV}} = 1 - H(\lambda_1, \lambda_2, \lambda_3, \lambda_4) + H\left(\frac{1+C}{2}, \frac{1-C}{2}\right), \quad (53)$$

where C is the maximum of $(|f_1|, |f_2|, |f_3|)$. In accordance with the assertion in Eq.(47), this is the same as the expression for symmetric discord derived in [6].

Several states, like Werner state, and others for which analytic results for various discords are available, fall in the category of vanishing average directions of both the spins [6]. The Q_{LHV} for such states correspond to special cases of (53).

2. Next, consider the following form of (48) for which analytic results for quantum discord are known [9]-[12],

$$\hat{\rho}^{\text{AB}} = \frac{1}{4}I^{\text{A}} \otimes I^{\text{B}} + \sum_{i=1}^3 c_i \hat{S}_i^{\text{A}} \otimes \hat{S}_i^{\text{B}} + \frac{r}{2} \hat{S}_3^{\text{A}} \otimes I^{\text{B}} + \frac{s}{2} I^{\text{A}} \otimes \hat{S}_3^{\text{B}}. \quad (54)$$

In this case

$$\langle \hat{\mathbf{S}}^{\text{A}} \rangle = \frac{r}{2} \mathbf{e}_3 = \langle \hat{\mathbf{S}}^{\text{B}} \rangle = \frac{s}{2} \mathbf{e}_3. \quad (55)$$

Hence, if $r \neq 0$, $s \neq 0$, $I_{\text{LHV}} = I(\mathbf{a}, \mathbf{b})$ with $\mathbf{a} = \mathbf{b} = \mathbf{e}_3$. The corresponding probabilities are given by

$$p(\epsilon^{\text{A}}, \epsilon^{\text{B}}) = \frac{1}{4} [1 + r\epsilon^{\text{A}} + s\epsilon^{\text{B}} + c_3\epsilon^{\text{A}}\epsilon^{\text{B}}]. \quad (56)$$

Using this, and the analytic expression for $I_Q(\hat{\rho}^{AB})$ [12], the LHV theoretic quantumness Q_{LHV} can be easily evaluated. In accordance with (34), it is same as the measurement induced disturbance.

In case, say, $r = 0$, the I_{LHV} is found by maximizing $I(\mathbf{a}, \mathbf{e}_3)$ over all \mathbf{a} . It is found that \mathbf{a} for which maxima is achieved corresponds to $\mathbf{a} = \mathbf{e}_3$. Hence, the appropriate LHV theoretic probability for computing Q_{LHV} in this case is given by (56) with $r = 0$. From the discussion circa (44), it follows that Q_{LHV} will be same as quantum discord Q_D for measurement over A if the direction of optimal measurement \mathbf{a}_m is \mathbf{e}_3 and $|\pm \mathbf{e}_3\rangle$ is the eigenbasis of $\hat{\rho}_{\mu\mu}^B(\mathbf{a}_m)$. In order to find the conditions under which the said equality is achieved, we recall that, assuming $|c_1| \geq |c_2|$, it has been shown in [9] that the direction \mathbf{a}_m of optimal measurements on A for quantum discord is \mathbf{e}_3 or \mathbf{e}_1 . The results of [9] have been qualified in [10] by deriving conditions under which \mathbf{a}_m is \mathbf{e}_1 or \mathbf{e}_3 or none of the two. On invoking those results, the condition under which $\mathbf{a}_m = \mathbf{e}_3$ in the present case of $r = 0$ reads $c_1^2 + r^2 \leq c_3^2$ and $|\pm \mathbf{e}_3\rangle$ constitutes eigenbasis of $\hat{\rho}_{\mu\mu}^B(\mathbf{a}_m)$. Hence, under the said condition, the LHV theoretic quantumness of mutual information and the quantum discord are identical.

Lastly, the case of $r = s = 0$ is the special case of (51) corresponding to $w_{ij} = c_i \delta_{ij}$.

4 Conclusions

By invoking explicitly the local hidden variable theory, a measure of quantumness of mutual information Q_{LHV} for a system of two spin-1/2 particles is proposed. It is based on finding the difference between the quantum and classical mutual informations in which the classical mutual information corresponds to the joint probability of the eigenvalues of the spins each along a specified direction. The proposed measure circumvents the need of optimization when the Bloch vector of each spin is non-zero; the optimization is needed but can be performed analytically exactly when the Bloch vector of each spin vanishes and is simplified when the Bloch vector of only one of the spins is zero. In essence, the proposed measure is identical with the measurement induced disturbance when the Bloch vector of each of the spins is non-zero. However, whereas the measurement induced disturbance is non-unique when the Bloch vector of one or both the spins is zero, the proposed measure even then determines the quantumness of mutual information unambiguously. The Q_{LHV} is identical with the symmetric discord if the Bloch vector of each spin vanishes. It is same as the quantum discord if the Bloch vector of only one spin is zero and if the state in question possesses certain additional properties.

References

- [1] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, Rev. Mod. Phys. **84**, 1655 (2012).
- [2] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, Rev. Mod. Phys. **A 81**, 865 (2009).
- [3] H. O. Olivier, W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2002).
- [4] M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, B. Synak-Radtke, Phys. Rev. **A 71**, 062307 (2005).
- [5] S. Luo, Phys. Rev. **A 77**, 022301 (2008).
- [6] S. Wu, U.V. Poulsen, and K. Molmer, 2009, Phys. Rev. **A 80**, 032319 (2009).
- [7] S. Luo, Phys. Rev. **A 77**, 042303 (2008).
- [8] X. M. Lu, J. Ma, Z. Xi, and X. Wang, Phys. Rev. **A 83**, 012327 (2011).
- [9] M. Ali, A. R. P. Rau, and G. Alber, Phys. Rev. **A 81**, 042105 (2010); see also Erratum, Phys. Rev. **A 82**, 069902(E) (2010).
- [10] Q. Chen, C. Zhang, S. Yu, X.X. Yi, and C. H. Oh, Phys. Rev. **A84**, 042313 (2011).
- [11] D. Girolami, and G. Adesso, Phys. Rev. **A 83**, 052108 (2011); *ibid* Phys. Rev. **A 84**, 052110 (2011).
- [12] M.S. Sarandy, Phys Rev. **A80**, 022108 (2009).
- [13] R.R.Puri, Phys. Rev. **A86**, 052111 (2012).
- [14] R.R.Puri, in 13th Asian Quantum Information Science Conference, The Institute of Mathematical Sciences, Chennai, India, 2013.
- [15] A. Streltsov, H.Kampermann, and D.Bruss, Phys. Rev. Lett. **106**, 160401 (2011).